

ON THE EXISTENCE OF NON-ABELIAN MONOPOLES: THE ALGEBRO-GEOMETRIC APPROACH

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ABSTRACT. We develop the Atiyah-Drinfeld-Manin-Hitchin-Nahm construction to study $SU(2)$ non-abelian charge 3 monopoles within the algebro-geometric method. The method starts with finding an algebraic curve, the monopole spectral curve, subject to Hitchin's constraints. We take as the monopole curve the genus four curve that admits a C_3 symmetry, $\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0$, with real parameters α , β and γ . In the case $\alpha = 0$ we prove that the only suitable values of γ/β are $\pm 5\sqrt{2}$ (β is given below) which corresponds to the tetrahedrally symmetric solution. We then extend this result by continuity to non-zero values of the parameter α and find finally a *new* one-parameter family of monopole curves with C_3 symmetry.

1. What is the monopole? Non-abelian monopoles appear naturally as particular solutions within the Standard Model, see e.g. the recent review [WY06] and the monograph [MS04]. The associated Lagrangian density in Minkowski space is given by

$$(1) \quad L = -\frac{1}{4}\text{Tr } F_{ij}F^{ij} + \frac{1}{2}\text{Tr } D_i\Phi D^i\Phi + V.$$

Here F_{ij} is the Yang-Mills field strength,

$$(2) \quad F_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j],$$

a_j the corresponding gauge field, D_i the associated covariant derivative acting on the Higgs field Φ by

$$D_i\Phi = \partial_i\Phi + [a_i, \Phi],$$

and V a potential. The gauge and Higgs fields take values in the Lie algebra of the gauge group. Static finite energy solutions of the Model are supposed to satisfy to the boundary conditions

$$(3) \quad \left. \sqrt{-\frac{1}{2}\text{Tr } \Phi(r)^2} \right|_{r \rightarrow \infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad \text{with } r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The positive integer $n \in \mathbb{N}$ in (3) is the first Chern number or the charge. Such a solution is called a *non-abelian monopole of charge n* .

We consider here non-abelian monopoles in the BPS (Bogomolny-Prasad-Sommerfeld) limit for which the potential $V = 0$ but the boundary conditions (3) remain preserved. Thus the configurations that minimize the energy of the system solve the *Bogomolny equations*

$$(4) \quad D_i\Phi = \pm \sum_{j,k=1}^3 \epsilon_{ijk} F_{jk}, \quad i = 1, 2, 3.$$

Moreover we fix the gauge group to be $SU(2)$: therefore our development deals with the *static $SU(2)$ monopoles in the BPS limit*.

2. The Atiyah-Drinfeld-Manin-Hitchin-Nahm construction. Although the Bogomolny equation is a first order partial differential equation in \mathbb{R}^3 few explicit solutions

are known for $n > 1$. Our results are based on the ADMHN construction that reduces this partial differential equation to a completely integrable ordinary differential equation. We summarize the construction in the form of the following theorem.

Theorem 1 (ADMHN). *The $su(2)$ charge n monopole solution is given by*

$$\begin{aligned}\Phi(\mathbf{x})_{\mu\nu} &= \imath \int_0^2 s \mathbf{v}_\mu^\dagger(\mathbf{x}, s) \mathbf{v}_\nu(\mathbf{x}, s) ds, \\ \alpha_i(\mathbf{x})_{\mu\nu} &= \imath \int_0^2 \mathbf{v}_\mu^\dagger(\mathbf{x}, s) \frac{\partial}{\partial x_i} \mathbf{v}_\nu(\mathbf{x}, s) ds, \quad i = 1, 2, 3,\end{aligned}\quad \mu, \nu = 1, 2.$$

where $\mathbf{v}_\mu(\mathbf{x}, s)$ are two orthonormalizable

$$\int_0^2 \mathbf{v}_\mu^\dagger(\mathbf{x}, s) \mathbf{v}_\nu(\mathbf{x}, s) ds = \delta_{\mu\nu}, \quad \mu, \nu = 1, 2,$$

solutions to the Weyl equation

$$(5) \quad \left(-\imath 1_{2n} \frac{d}{ds} + \sum_{j=1}^3 (T_j(s) + \imath x_j 1_n) \otimes \sigma_j \right) \mathbf{v}(\mathbf{x}, s) = 0.$$

The $n \times n$ matrices $T_j(s)$, $s \in (0, 2)$, called Nahm data, satisfy the Nahm equations

$$(6) \quad \frac{dT_i(s)}{ds} = \frac{1}{2} \sum_{j,k=1}^3 \epsilon_{ijk} [T_j(s), T_k(s)].$$

The residues $\text{Res}_{s=0} T_i(s)$ and $\text{Res}_{s=2} T_i(s)$ form irreducible n -dimensional representations of $su(2)$. Also the following hermiticity conditions are satisfied

$$T_i(s) = -T_i^\dagger(s), \quad T_i(s) = T_i^\dagger(2-s).$$

3. The Hitchin construction. The complete integrability of Nahm's equations (6) was proved by Hitchin in [Hit82]. These equations can be written the Lax form,

$$(7) \quad \frac{dA(s, \zeta)}{ds} = [A(s, \zeta), M(s, \zeta)],$$

where ζ is a spectral parameter and $A(s, \zeta)$, $M(s, \zeta)$ are $n \times n$ matrices

$$\begin{aligned}A(s, \zeta) &= A_{-1}(s) \zeta^{-1} + A_0(s) + A_{+1}(s) \zeta, \\ M(s, \zeta) &= \frac{1}{2} A_0(s) + \zeta A_{+1}(s), \\ A_{\pm 1}(s) &= T_1(s) \pm \imath T_2(s), \quad A_0(s) = 2\imath T_3(s).\end{aligned}$$

The known consequence of the Lax representation is that the equation

$$(8) \quad \det(A(s, \zeta) - \eta 1_n) = 0$$

represents a polynomial in (η, ζ) with coefficients independent of s . That is the spectral curve $\widehat{\mathcal{C}}$ is given by the equation

$$(9) \quad \eta^n + \alpha_1(\zeta) \eta^{n-1} + \dots + \alpha_n(\zeta) = 0,$$

where $\alpha_k(\zeta)$ are polynomials in ζ of degree not exceeding $2k$. The genus of $\widehat{\mathcal{C}}$ is generically

$$(10) \quad g_{\widehat{\mathcal{C}}} = (n-1)^2.$$

4. Theta-functions. Riemann's θ -function is a powerful instrument of analysis of algebraic curves and their Jacobians, see e.g. [Fay73]. θ -functions depend on two groups of variables, a complex vector $\mathbf{z} \in \mathbb{C}^g$ and a period matrix belonging to the Siegel upper half-space

$$(11) \quad \hat{\tau} : \quad \hat{\tau}^T = \hat{\tau}, \quad \Im(\hat{\tau}) > 0.$$

The period matrix $\hat{\tau}$ is built from a complete set of linearly independent holomorphic differentials

$$(12) \quad \mathbf{u}(\xi, \eta) = (u_1(\xi, \eta), \dots, u_g(\xi, \eta))^T$$

and a canonical homology basis

$$(13) \quad (\mathbf{a}_1, \dots, \mathbf{a}_g; \mathbf{b}_1, \dots, \mathbf{b}_g), \quad \mathbf{a}_i \circ \mathbf{b}_j = \delta_{i,j}, \quad \mathbf{a}_i \circ \mathbf{a}_j = \mathbf{b}_i \circ \mathbf{b}_j = 0.$$

Denoting the matrices of \mathbf{a} and \mathbf{b} -periods as

$$\mathcal{A} = \left(\oint_{\mathbf{a}_i} u_j(\xi, \eta) \right)_{i,j=1,\dots,g}, \quad \mathcal{B} = \left(\oint_{\mathbf{b}_i} u_j(\xi, \eta) \right)_{i,j=1,\dots,g}.$$

we then define $\hat{\tau} = \mathcal{B}\mathcal{A}^{-1}$. The θ -function of the algebraic curve $\hat{\mathcal{C}}$ is given by the Fourier series

$$(14) \quad \theta(\mathbf{z}; \hat{\tau}) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp \{ i\pi \mathbf{n}^T \hat{\tau} \mathbf{n} + 2i\pi \mathbf{z}^T \mathbf{n} \}.$$

θ -functions possesses periodicity properties when the argument is shifted by a period, and modular properties when the homology basis is mapped to another one. We do not present these well-known formulae here.

5. Hitchin's constraints. Not all curves of the form (9) can serve as a spectral curve of a monopole but only those that satisfy the *Hitchin constraints*, denoted below as **H1**, **H2** and **H3**. These constraints were formulated in [Hit82, Hit83] as conditions on the cohomology groups of holomorphic line bundles associated to the spectral curve. Here we will present these conditions in equivalent form by following to the Ercolani-Sinha paper [ES89] and our preprint [BE06] that is published in journal form in [BE10a] and [BE10b].

H1. The spectral curve $\hat{\mathcal{C}}$ admits the involution

$$(15) \quad (\zeta, \eta) \rightarrow \left(-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2 \right).$$

H2. The \mathbf{b} -periods of a normalized differential of the second kind $\gamma_\infty(P)$ are half-integer, where

$$(16) \quad \gamma_\infty(P)_{P \rightarrow \infty_i} = \left(\frac{\rho_i}{\xi^2} + O(1) \right) d\xi, \quad \rho_i = \lim_{\zeta \rightarrow \infty_i} \frac{\eta}{\zeta^2},$$

$$(17) \quad \oint_{\mathbf{a}_k} \gamma_\infty = 0, \quad k = 1, \dots, g,$$

$$(18) \quad \mathbf{U} = \frac{1}{2\pi i} \left(\oint_{\mathbf{b}_1} \gamma_\infty, \dots, \oint_{\mathbf{b}_g} \gamma_\infty \right)^T = \frac{1}{2} \mathbf{n} + \frac{1}{2} \tau \mathbf{m}.$$

Here the integer vectors $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^g$ are the *Ercolani-Sinha vectors* that were introduced in [ES89] and will play the role of the principal variables in this exposition.

As noted in [ES89], the constraint **H2** is a very restrictive condition on the moduli of the curve and *a priori* it is not clear if such a curve exists for $n > 2$. It places $g\hat{\mathcal{C}}$ real constraints on the coefficients of (9).

H3. The linear winding vector $Us + K$, where K is the vector of Riemann constants, does not intersect theta-divisor inside the interval $(0, 2)$:

$$(19) \quad \theta(Us + K; \hat{\tau}) \neq 0, \quad s \in (0, 2).$$

6. Existence of the tetrahedral monopole. We will restrict our analysis to the special class of curves that respect the C_3 symmetry,

$$(20) \quad \sigma : (\eta, \zeta) \longrightarrow (\rho\eta, \rho\zeta), \quad \rho = e^{2i\pi/3}.$$

This symmetry corresponds to a space-time symmetry of the monopole [HMM95]. The most general charge 3 monopole curve that admits such a C_3 symmetry and satisfies **H1** may be put in the form

$$(21) \quad \eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0,$$

where α, β, γ are real numbers. We start by considering an even more special subclass of C_3 symmetric curves, namely

$$(22) \quad \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0.$$

We report here the following

Theorem 2 (On the existence of tetrahedral monopole [BE10]). *The class of the monopole curves (22) contains the two representatives,*

$$(23) \quad b = \pm 5\sqrt{2}, \quad \chi = -\frac{1}{6} \frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}.$$

7. Demonstration. Our proof is based on various results, old and new.

7.1 Wellstein and Matsumoto. Consider the curve of genus $g = 4$

$$(24) \quad w^3 = (z - \lambda_1) \dots (z - \lambda_6), \quad \lambda_i \neq \lambda_j \in \mathbb{C}$$

The associated holomorphic differentials are

$$\frac{dz}{w}, \quad \frac{dz}{w^2}, \quad \frac{zdz}{w^2}, \quad \frac{z^2dz}{w^2}.$$

Let $\{\mathbf{a}_1, \dots, \mathbf{a}_4; \mathbf{b}_1, \dots, \mathbf{b}_4\}$ be the homology basis shown in Fig.1

Denote the vector of periods

$$\mathbf{X} = \left(\oint_{\mathbf{a}_1} \frac{dz}{w}, \dots, \oint_{\mathbf{a}_4} \frac{dz}{w} \right)^T.$$

In 1899 Wellstein showed [Wel899] that the period matrix $\hat{\tau}$ is of the form

$$(25) \quad \hat{\tau} = \rho^2 \left(H + (\rho^2 - 1) \frac{H\mathbf{X}\mathbf{X}^T H}{\mathbf{X}^T H \mathbf{X}} \right),$$

where $\rho = \exp(2i\pi/3)$, $H = \text{diag}(1, 1, 1, -1)$. This was rediscovered by Matsumoto in 2000 [Mat00] and a further proof given in [BE06]. We will implement Wellstein's result in the case of the special curve of the form (21) given by (22). This is still of genus four.

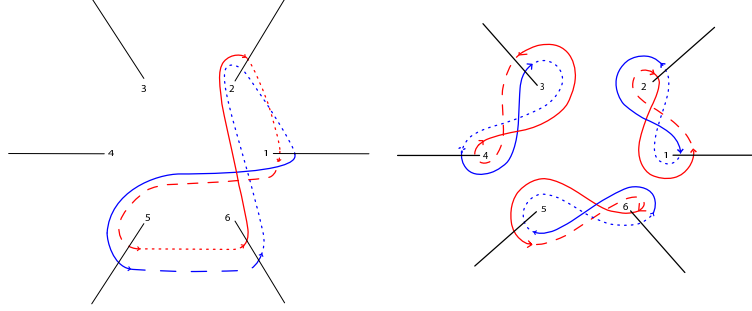


FIGURE 1. Homology basis of the genus four curve $w^3 = (z - \lambda_1) \dots (z - \lambda_6)$. The cuts connect three sheets and are connected at infinity. Arcs are depicted by solid lines on the first (upper) sheet, dotted lines on the second sheet and dashed lines on the third sheet

It was shown in [BE06] that for a pair of relatively prime integers (m, n) for which $(m + n)(m - 2n) < 0$ the following solution to **H1** and **H2** could be constructed. First one solves for t the equation involving hypergeometric functions

$$\frac{2n - m}{m + n} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, 1 - t\right)}.$$

Then values of parameters b and χ are given by

$$(26) \quad b = \frac{1 - 2t}{\sqrt{t(1 - t)}}, \quad \chi^{1/2} = -(n + m) \frac{2\pi}{3\sqrt{3}} \sqrt[3]{t^2(1 - t)} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, t\right).$$

The Ercolani-Sinha vectors are then expressible in terms of two integers $m, n \in \mathbb{Z}$,

$$(27) \quad \mathbf{n} = \begin{pmatrix} n \\ m - n \\ -m \\ 2n - m \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} -m \\ n \\ m - n \\ 3n \end{pmatrix}.$$

The period matrix then takes the form

$$(28) \quad \hat{\tau} = \rho^2 H + (\rho - \rho^2) \frac{(\mathbf{n} + \rho^2 H \mathbf{m})(\mathbf{n} + \rho^2 H \mathbf{m})^T}{(\mathbf{n} + \rho^2 H \mathbf{m})^T H (\mathbf{n} + \rho^2 H \mathbf{m})}.$$

i.e. it depends on the integers (m, n) and root of unity ρ .

7.2 A Strange equation. Comparing our parametrization with Hitchin, Manton and Murray's tetrahedral solution [HMM95] we conclude that with $n = 1$ and $m = 0$ we should have

$$(29) \quad \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)} = 2,$$

$$(30) \quad t = \frac{1}{2} + \frac{5\sqrt{3}}{18}, \quad b = -5\sqrt{2}.$$

One can see that although equation (29) is transcendental it is nonetheless solved in radicals (30). There is also a solution which corresponds to a physical inversion of the $n = 1, m = 0$ solution with $n = m = 1$: for this latter solution $t = \frac{1}{2} - \frac{5\sqrt{3}}{18}$, and $b = 5\sqrt{2}$. We will focus on the former solution in the ensuing discussion.

The following question arises: can one find numbers t such that

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)} = \frac{2n-m}{n+m} \in \mathbb{Q}?$$

Each such solution will then provide a curve satisfying **H1** and **H2**.

7.3 Ramanujan's hypergeometric relation. The answer to the question just posed follows from Ramanujan's hypergeometric relation presented in the Second Notebook [Ber98].

Let r (the signature) and $n \in \mathbb{N}$. Then the following hypergeometric equality holds when x, y are the zeros of a (necessarily) algebraic equation $\mathcal{P}(x, y) = 0$,

$$(31) \quad \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)} = n \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-y\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; y\right)}.$$

A consequence of this is that the numbers t above are *algebraic*. Ramanujan found this equation for signature $r = 3$ and $n = 2$ where

$$(32) \quad (xy)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}(1-y)^{\frac{1}{3}} = 1.$$

Setting $y = \frac{1}{2}$ in (32) we obtain $x = \frac{1}{2} \mp \frac{5\sqrt{3}}{18}$ and $b = \pm 5\sqrt{2}$. Therefore Ramanujan's relation stands behind the existence of the tetrahedral monopole!

To complete the proof of the existence of a monopole spectral curve it remains to check that the curves satisfying **H1** and **H2** also satisfy **H3**, i.e. to show that for $s \in (0, 2)$ the winding vector does not intersect the θ -divisor. To the best knowledge of the authors there are no analytic methods to check this condition and we are only able to check **H3** numerically. To do that we plot the real and imaginary part of the the function of the variable s , $\theta(\mathbf{U}s + \mathbf{K})$, where \mathbf{U} is the Ercolani-Sinha vector associated to the tetrahedron, i.e. $n = 1, m = 0$. The plot shown in Fig. 2 confirms the validity of the condition **H3** for this curve.

8. Uniqueness of the tetrahedral monopole. Using Ramanujan's hypergeometric relations many other solutions of (31) were found [BBG95] and from each of these one may construct curves satisfying the constraints **H1** and **H2**. Despite numerous attempts to find values for the Ercolani-Sinha vectors different from the tetrahedrally symmetric case just described no new solutions satisfying **H3** have been found. We have conjectured that the solution corresponding to arbitrary n, m has $2(|n| - 1)$ unwanted zeros in the interval $s \in (0, 2)$. For example, in the case $n = 4, m = -1$, the plot of $|\theta(\mathbf{U}s + \mathbf{K})|$ is given in Fig.3 which shows 6 unwanted zeros. Therefore the corresponding value of the parameter b does

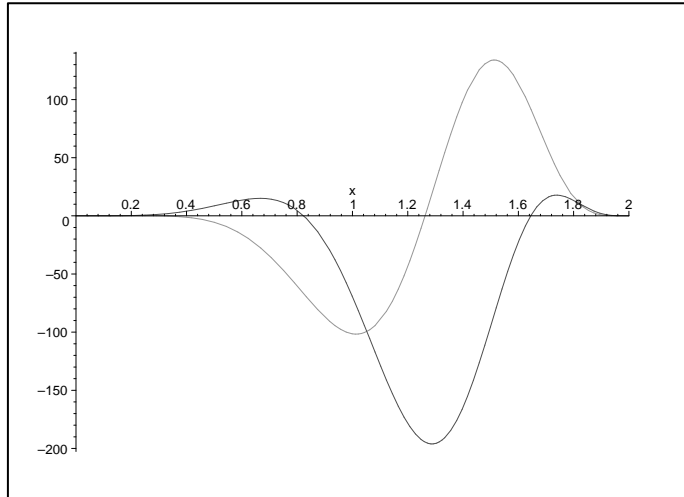


FIGURE 2. Plot of the real and imaginary parts of the function $\theta(Us + \mathbf{K})$ for the case $n = 1, m = 0$

not lead to a monopole curve. Although unable to prove this general conjecture we are able to prove the following theorem.

Theorem 3 (The uniqueness of the tetrahedral monopole [BE10]). *The class of the monopole curves (22) consists of only two representatives,*

$$(33) \quad b = \pm 5\sqrt{2}, \quad \chi = -\frac{1}{6} \frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}.$$

In other words there are no monopoles associated to the curve (22) beyond those with tetrahedral symmetry.

This is a statement for all integers $(m, n) \in \mathbb{Z}^2$. Clearly this cannot be proven by resort to plots. We will demonstrate below the problem can be reduced to the analysis of certain one-dimensional subsets in the plane. We are able to do this by implementing one of the most remarkable achievements of the theory of θ -functions, namely the Schottky-Jung proportionalities [Fay73].

8.1 Schottky-Jung proportionality. Schottky-Jung theory permits the reduction of θ -functions to θ -functions of lower genera for certain subspaces of the Jacobi variety when a curve admits coverings. We consider here the case of an unramified cover which may be associated to our family of curves (20). Indeed our genus 4 curve $\hat{\mathcal{C}}$ covers 3-sheetedly a

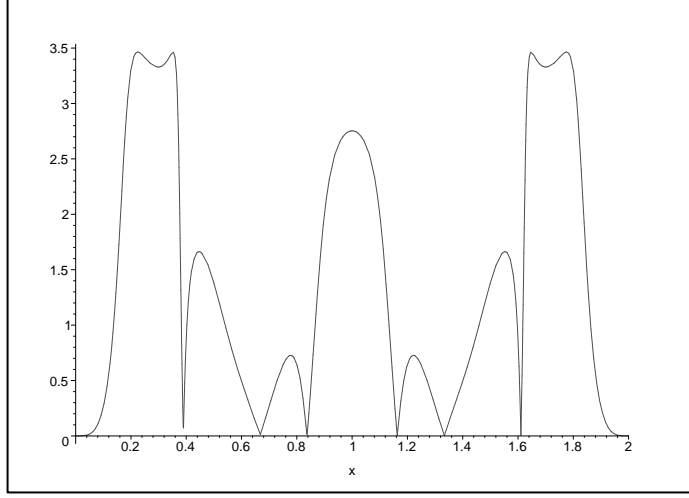


FIGURE 3. Plot of the absolute value $|\theta(\mathbf{U}x + \mathbf{K})|$ for the case $n = 4$, $m = -1$.

genus 2 curve \mathcal{C} : $\pi : \hat{\mathcal{C}} \rightarrow \mathcal{C}$ with

$$(34) \quad \hat{\mathcal{C}} : \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0,$$

$$(35) \quad \mathcal{C} : \nu^2 = (\mu^3 + b)^2 + 4,$$

and $\nu = \zeta^3 + 1/\zeta^3$, $\mu = -\eta/\zeta$. The Riemann-Hurwitz formula,

$$2 - 2\hat{g} = B + N(2 - g)$$

shows that the cover π is unramified, i.e. $B = 0$. More generally one can associate to a class of C_n symmetric curves an n -sheeted unbranched cover of a hyperelliptic curve of genus $n - 1$ that is the spectral curve for the $su(n)$ affine Toda theory [Bra10].

According to the Schottky-Jung theory (we are following here [Fay73]) in the case of an unramified cover there exists a basis in the homology group

$$H_1(\hat{\mathcal{C}}, \mathbb{Z}) \ni (\mathbf{a}_1, \dots, \mathbf{a}_4; \mathbf{b}_1, \dots, \mathbf{b}_4)$$

admitting the automorphism σ such that,

$$(36) \quad \begin{aligned} \sigma \circ \mathbf{a}_k &= \mathbf{a}_{k+1}, & \sigma \circ \mathbf{b}_k &= \mathbf{b}_{k+1}, & k &= 1, 2, 3 \\ \sigma \circ \mathbf{b}_0 &= \mathbf{b}_0, & \sigma \circ \mathbf{a}_0 &\sim \mathbf{a}_0, \end{aligned}$$

(where \sim means ‘homologous to’). The period matrices of the curves $\hat{\mathcal{C}}$ and \mathcal{C} are related by

$$(37) \quad \hat{\tau} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}, \quad \tau = \begin{pmatrix} \frac{1}{3}a & b \\ b & c + 2d \end{pmatrix}.$$

Remarkably under these conditions the following θ -function factorization occurs

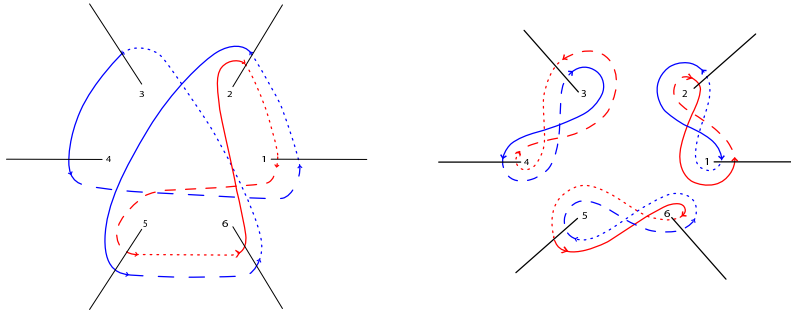


FIGURE 4. Symmetric homology basis of the curve (22)

Theorem 4 (The Fay-Accola theorem in the case of $g = 4$). *In the case of the genus $g = 4$ 3-sheeted unramified covering of the genus two curve the theta-factorization has the form*

$$(38) \quad \frac{\theta(3z_1, z_2, z_2, z_2; \hat{\tau})}{\theta(z_1, z_2; \tau)\theta(z_1 + 1/3, z_2; \tau)\theta(z_1 - 1/3, z_2; \tau)} = \kappa,$$

where $\hat{\tau}$ and τ are given in (37) and κ is independent of \mathbf{z} .

We remark that the Fay-Accola theorem depends strongly on the pull-back formula $(z_1, z_2) \rightarrow (3z_1, z_2, z_2, z_2)$, but the \mathbf{z} -argument of the genus four θ -function has the necessary form.

8.2 Homology transformation. To implement the Fay-Accola theorem we should first find a ‘cyclic’ homology basis (36) for the curve (21). Such a basis is given in Fig.4. Further, when the curve (21) is reduced to (22) we wish to know the symplectic transformation between the homology basis given on the Fig.1 and that of Fig.4. This will permit us to compare results obtained for this basis with the previous ones of [BE06].

The cyclic homology basis was found in [D’Av10, BDE10] using software developed by Northover [Nor10, Nor10a]. This software also provides us with the desired symplectic transformation [BN09]. Passing in our formulae for θ -functions to the cyclic homology basis we are able to reduce the analysis of the vanishing of the genus four θ -function to the analysis of the vanishing of three genus two θ -functions with arguments shifted by $\pm \frac{1}{3}$, as is evident in (38).

8.3 Humbert variety. Each of the aforementioned genus two θ -functions admits a further reduction to elliptic θ -functions. This is because the period matrix τ matrix belongs to the so called *Humbert variety* that is defined as follows. The Humbert variety \mathcal{H}_Δ consists

of those period matrices τ of a genus two curve \mathcal{C} in the Siegel upper half-space that satisfy

$$(39) \quad \begin{aligned} q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) &= 0; \\ q_i \in \mathbb{Z}, \quad \Delta &= q_3^2 - 4(q_1q_5 + q_2q_4). \end{aligned}$$

It is known that in the case $\Delta = h^2$, $h \in \mathbb{N}$, there exists a symplectic transformation \mathfrak{S} that reduces the period matrix to the quasi-diagonal form

$$\mathfrak{S} : \tau \rightarrow \mathfrak{S} \circ \tau = \begin{pmatrix} T_1 & \frac{1}{h} \\ \frac{1}{h} & T_2 \end{pmatrix}, \quad h \in \mathbb{N}.$$

The integer h is the degree of the cover \mathcal{C} over two elliptic curves $\mathcal{E}, \mathcal{E}'$

$$\mathcal{E}' \leftarrow \mathcal{C} \rightarrow \mathcal{E}.$$

In the case we are considering the associated genus two curve is a two-sheeted cover over an elliptic curve, i.e. $h = 2$. The underlying genus two curve has D_6 symmetry and is given by Bolza's classification of genus two curves with many automorphisms [Bol887]. Thus the genus two period matrix appearing in our study is equivalent to

$$(40) \quad \begin{pmatrix} T & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{12T} \end{pmatrix}$$

where the complex variable T is expressible in terms of the Ercolani-Sinha vector and roots of unity. Thus a complete reduction of the initial genus four θ -function to elliptic θ 's occurs.

Proposition 5 (On the H3 condition [BE10]). *The vanishing of the genus four θ -function*

$$\theta(\mathbf{U}s + \mathbf{K}; \hat{\tau}) = 0 \quad \text{for } s \in (0, 2)$$

*of the curve $\hat{\mathcal{C}}$ given in (22) and satisfying **H1** and **H2** occurs if and only if one of the following three equalities is satisfied*

$$(41) \quad \frac{\vartheta_3}{\vartheta_2} \left(y\sqrt{-3} + \varepsilon \frac{T}{3} |T \right) + (-1)^\varepsilon \frac{\vartheta_2}{\vartheta_3} \left(y + \varepsilon \frac{1}{3} | \frac{T}{3} \right) = 0,$$

where $\varepsilon = 0, \pm 1$, and

$$y = \frac{1}{3}s(n+m), \quad T = \frac{2\sqrt{-3}(n+m)}{2n-m}.$$

Therefore the function $y = y(T)$ implicitly defined by (41) provides the answer to the question of whether or not **H3** satisfied. In studying this equation we found (new?) θ -constant relations

$$\frac{\vartheta_3}{\vartheta_2} \left(\frac{T}{3} | T \right) = \frac{\vartheta_2}{\vartheta_3} \left(\frac{1}{3} | \frac{T}{3} \right)$$

and

$$\vartheta_4^2(0|T) \imath \sqrt{3} \frac{\vartheta_1(\frac{T}{3}|T) \vartheta_4(\frac{T}{3}|T)}{\vartheta_2^2(\frac{T}{3}|T)} + \vartheta_4^2(0|\frac{T}{3}) \imath \sqrt{3} \frac{\vartheta_1(\frac{1}{3}|\frac{T}{3}) \vartheta_4(\frac{1}{3}|\frac{T}{3})}{\vartheta_3^2(\frac{1}{3}|\frac{T}{3})} = 0.$$

Both relations can be proven by using Ramanujan's parametrization of the Jacobian moduli of elliptic curves whose periods are T and $T/3$, see [Law89] and [BBG95]. The above θ -constant relations are used to analyze the plot in Fig.5. They show that only in the two cases, when $(n+m)/(2n-m) = 2$ and $(n+m)/(2n-m) = 1/2$ corresponding to the factors of 2 and 1/2 in the Ramanujan's hypergeometric relation (31), does the θ -divisor only intersect the boundaries of the segment $[0, 2]$ and no interior points. Therefore we can conclude that no charge 3 monopoles exist for this class of curves beyond those with tetrahedral symmetry.

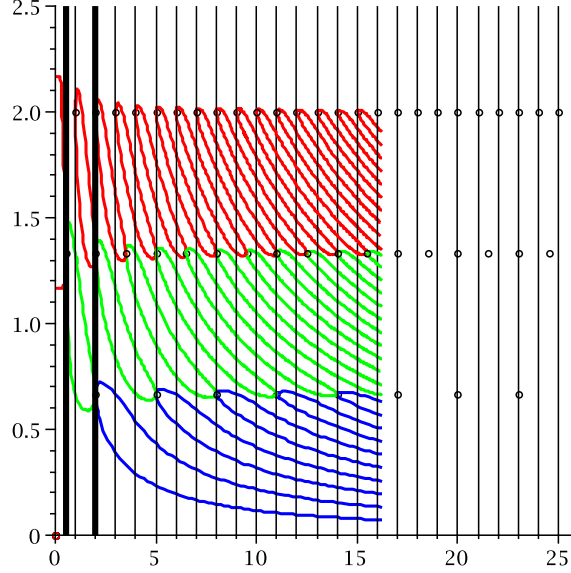


FIGURE 5. Three branches of the function $y(T)$ where y/ρ is plotted against $(n+m)/(2n-m)$. Here $T = 2\sqrt{-3}(n+m)/(2n-m)$.

9. A new monopole curve. Being armed with the tetrahedral solution we are able to extend this result to the general charge three monopole curve with C_3 symmetry given by formula (21). First, by rescaling of the variables to $(a, g) := (\alpha/\beta^{2/3}, \gamma/\beta)$, one may recast the Ercolani-Sinha constraints to finding the (a, g) such

$$(42) \quad 0 = \oint_{\mathfrak{c}} \frac{dX}{Y}, \quad Y^2 = (X^3 + aX + g)^2 + 4,$$

for a cycle \mathfrak{c} specified by the solutions $n = 1, m = 0$ and $n = 1, m = 1$. The remaining Ercolani-Sinha constraint simply determines β in terms of (a, g) . Thus starting with the points $(a, g) = (0, 5\sqrt{2})$ and $(a, g) = (0, -5\sqrt{2})$ we can find the line in the real (a, g) -plane along which **H1**, **H2**, **H3** are satisfied.

That is one of the results of A.D'Avanzo [D'Av10] and [BDE10]. The outline of the method is as follows. The integral is evaluated using the genus two arithmetic-geometric mean (AGM) which generalizes the Gaussian AGM method for calculating complete elliptic integrals of the first kind. The genus two AGM method as presented in the Bost-Mestre article [BM88] deals mainly with real branch points and a modification of this method to the case of a real curve and complex branch points was developed. Using this AGM we may quickly determine those a and g for which (42) is satisfied.

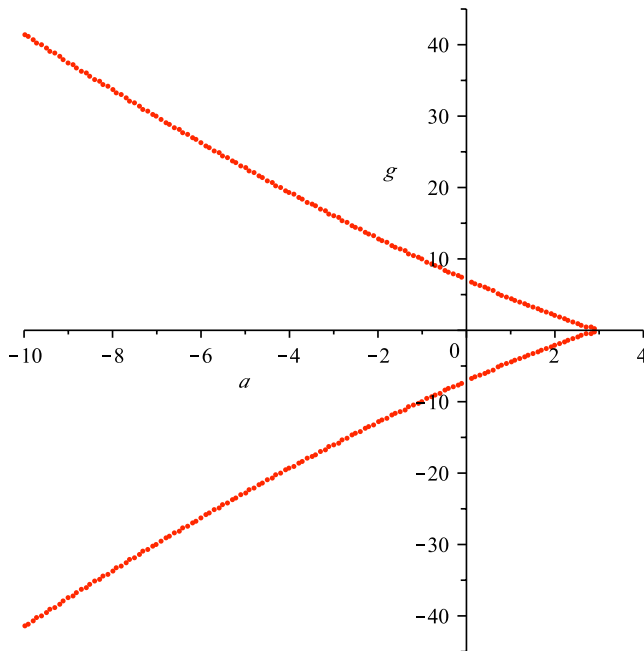


FIGURE 6. Solutions to the Ercolani-Sinha constraints

The corresponding plot is given in Fig.6. This curve reproduces the asymptotic behavior predicted by [HMM95]. One can see that the cusp point $(3, 0)$ appears on the plot. The curve here reduces to the rational curve

$$(43) \quad y^2 = (x^2 + 4)(x^2 + 1)^2,$$

i.e. becomes singular. We remark that this behavior of the solution curve is consistent with Sutcliffe's prediction that the curve (43) describes a configuration of three unit-charge monopoles with dihedral D_3 symmetry, constituting an asymptotic state for a 3-monopole configuration (cf. eq.(4.16) in [Sut97]).

10. Discussion. In this note we have concentrated on the finding the algebraic curve that satisfies Hitchin's constraints for the spectral curve of monopole. We succeeded in finding a new one-parameter family of charge 3 monopole curves of the form $\eta^3 + \alpha\eta\zeta^2 + \beta\zeta^6 + \gamma\zeta^3 - \beta = 0$ that admits C_3 symmetry.

But this is only the first step of the construction. The ultimate aim is to find the Higgs field Φ and gauge fields a_i in closed analytic form. To the best knowledge of the authors, with the exception of the $n = 1$ and $n = 2$ axially symmetric cases, no such analytic expressions have yet been found. Our program is to calculate such analytic quantities and the work presented here is part of that program. Knowledge of the monopole curve allows one to develop the algebro-geometric integration of the Nahm equation as was done in [ES89]. These last results were improved in [BE06]. A common perception is that the ADHMN construction requires the numerical solution of the Weyl equations with potentials provided by the Nahm data. We are seeking the θ -functional integration of the Weyl equations themselves. In [BE09] we use an ansatz of Nahm [Nah82] which reduces the integration of

the $2n$ -th order system of ODE of the Weyl equations to an n -th order ODE system that is equivalent to the linear spectral problem of the Lax representation for Nahm equation. Therefore the algebro-geometric solution to the Weyl equation is given in terms of a Baker-Akhiezer function of the Nahm equation whose spectral parameter is a function of monopole coordinates. We believe that such a program is realizable for higher charge monopoles and, in particular, for the one-parameter family of trigonal curves reported here. This approach in the case of a non-axially symmetric charge 2 monopole is now the focus of our attention and the results will be published elsewhere.

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